JULIA SETS OF RELAXED NEWTON'S METHOD FOR POLYNOMIALS WITH TWO DISTINCT ROOTS

A Project Report Submitted
in Partial Fulfilment of the Requirements
for the Degree of

Joint M.Sc-Ph.D

in

Mathematics

by

SATYABRATA JANA

(Roll No.-13MA05017)

Under the supervision of

Dr. TARAKANTA NAYAK

to the

SCHOOL OF BASIC SCIENCES

INDIAN INSTITUTE OF TECHNOLOGY BHUBANESWAR

BHUBANESWAR - 751 007, INDIA

May 2015

APPROVAL OF THE VIVA-VOCE BOARD

Certified that the Project report entitled "Julia sets of relaxed Newton's method for polynomials with two distinct roots" submitted by Satyabrata Jana to the Indian Institute of Technology, Bhubaneswar for the award of the degree "Master in Science" is accepted by the External Examiner and the student has successfully defended the Project report in the viva-voce examination held on 05/05/2015.

(Supervisor)

(Internal Examiner 1) (Internal Examiner 2)

CERTIFICATE

This is to certify that the project report entitled "**Julia sets of relaxed Newton's method for polynomials with two distinct roots**" submitted by **Satyabrata Jana** (Roll. No.-13MA05017) to the school of Basic Sciences, Indian Institute of Technology Bhubaneswar, towards the requirement of the course MA5502 Project , is a record of bonafide research work under my supervision and I consider it worthy of consideration for the award of the degree of "**Master of Science**" of the Institute.

May, 2015
Bhubaneswar

Dr. Tarakanta Nayak
Project Supervisor
School of Basic Sciences
IIT Bhubaneswar
Bhubaneswar - 751 007

DECLARATION

I certify that

- The work contained in the report is original and has been done by myself and under general supervision of my project guide.
- This work has not been submitted to any other Institute for any degree or diploma.
- I have followed the guidelines provided by the Institute in writing the project.
- I have confirmed to the norms and guidelines given in the Ethical Code of Conduct of the Institute.
- Whenever I have used materials (text, theoretical analysis and data) from other sources, I have given due credit to them by citing them in the text of the project and giving their details in the references.
- Whenever I have quoted written materials from other sources, I have put them under quotation and given due credit to the source by citing them and by giving required details in the references.

May, 2015 Signature of the Student

Bhubaneswar

ACKNOWLEDGMENTS

Doing this project work is one of the most wonderful experiences to me. It is not only about writing a report and presenting it, but also about learning, discussing and accepting suggestion to improve foremost. I would like to express my sincere gratitude and respect to my project supervisor **Dr. Tarakanta Nayak** for his guidance, inspiration and constant encouragement throughout my project work. I wish to express my sincere thanks to **Prof. Saroj Nayak**, Head of the School of Basic Sciences for the facilities and infrastructures and all the faculty members for their moral cooperation. I would also like to thank **Dr. Sabyasachi Pani**, faculty advisor, for his immense support and guidance. My heartfelt thanks to my friend Anudeep Nain who helped me when needed, also thanks to my family members for their love and support.

5 May 2015 (Satyabrata Jana)

Bhubaneswar

Abstract

Finding the roots of a function is a fundamental problem in science and engineering. Newton's method is one of the easiest iterative root-finding algorithm. This method is very efficient near a simple root because the algorithm converges quadratically in the neighborhood of such a root. But at a multiple root Newton's method converges linearly. That's why various modifications of this method have been proposed which converges quadratically at multiple roots. Newton's method defines a dynamical system on the Riemann sphere.we define and describe few properties of Newton's method. We consider polynomial with two roots of multiplicity m and n. It is shown that the relaxed Newton's method N_m is conjugate to a quadratic polynomial. Various examples are discussed and pictures of basin of attractions are made using matlab.

Contents

List of Figures

Chapter 1

Introduction

One of the best known iteration functions for finding a root of a complex valued function of a complex variable is Newton's method. It is useful to think of the iteration function N as being defined on the whole Riemann sphere i.e. the complex numbers with the point at infinity adjoined to it. The global study of Newton's method can be analyzed using the theory of complex dynamics of rational maps on the Riemann sphere [1].

1.1 The standard Newton's method

Let $f(z)$ be a polynomial, Then the rational iteration function

$$N(z) = z - \frac{f(z)}{f'(z)}$$

is called standard Newton's method for $f(z)$. If $f'(z) \neq 0$ for any $z \in \mathbb{C}$, $f(z^\star) = 0$ if and only if $N(z^\star) = z^\star$, i.e. $z^\star$ is fixed point for $N(z)$. However, standard Newton's method has some shortcomings in root finding. When the order of root is greater than one, this method converges only linearly(order of convergence is defined in Section 1.4. Various modifications of Newton's method have been proposed which converge quadratically at least at multiple roots. Two such method are Relaxed Newton's method and Newton's method for multiple roots. These methods converge quadratically even when the order of the root is greater than one.

1.2 The Relaxed Newton's method

Let $f(z)$ be polynomial. The relaxed Newton's method of order k is defined as

$$N_k(z) = z - k\frac{f(z)}{f'(z)}.$$

Relaxed Newton's method has the advantage of converging quadratically at least. But it has also limitation. In order that N_k converge quadratically to a multiple root, k is to be the multiplicity of the root. So the order of that root should be known before applying this method.

1.3 Newton's method for multiple roots

Let $f(z)$ be a polynomial and $z^\star$ is a multiple root of $f(z)$. Then $u(z) = 0$ has a simple root at $z^\star$ where $u(z) = \frac{f(z)}{f'(z)}$. Applying the standard Newton's method to $u(z) = 0$ we obtain

$$F(z) = z - \frac{u(z)}{u'(z)} = z - \frac{f(z)f'(z)}{[f'(z)]^2 - f''(z)f(z)}.$$

This method converges at least quadratically. Of course, computing second derivative needs more calculation.

1.4 Rate of convergence

In numerical analysis, the speed at which a convergent sequence approaches its limit is quantified in terms of the rate of convergence. A limit does not give information about the number of first few terms of the sequence. This concept of rate of convergence has practical importance, particularly when we deal with a sequence of successive approximations in an iterative method. Typically fewer iterations are needed to yield a useful approximation if the rate of convergence is higher. The sequence $\{x_k\}$ converges with order q to L for $q \geq 1$ if there exists a $\mu > 0$ such that

$$\lim_{k\to\infty} \frac{|x_{k+1} - L|}{|x_k - L|^q} = \mu.$$

In particular, when $q = 1$, and $\mu \in (0, 1)$, it is called linear convergence. For $q = 2$, it is called quadratic convergence, and for $q = 3$ it is called cubic convergence and so on.

1.5 Periodic Point

We call w, a periodic point of an iteration function $R(z)$ of period p if $R^p(w) = w$ and $R^i(w) \neq w$ for any $i < p$.

There are four type of periodic points. Let $(R^p)'(w) = \lambda$ then w is

$$\begin{cases} \text{Super attracting,} & \text{if } \mid \lambda \mid = 0; \\ \text{Attracting,} & \text{if } 0 < \mid \lambda \mid < 1; \\ \text{Indifferent or neutral or litmusless,} & \text{if } \mid \lambda \mid = 1; \\ \text{Repelling,} & \text{if } \mid \lambda \mid > 1. \end{cases}$$

When $p = 1$, w is called fixed a point of $R(z)$.

1.6 Basin of attraction

An iteration function is a function from some set to itself which is obtaining by composing that function certain number of times. The process of repeatedly applying the same function is called iteration. The Basin of attraction of a fixed point θ of a iteration function $R(z)$ is the set of all those points which converges to θ under $R(z)$. It is denoted by $A(\theta)$. So it is defined as

$$A(\theta) = \{z : \lim_{k \to \infty} R^k(z) = \theta\}.$$

For example let $f(z) = z^2$. 0 is a fixed point of $f(z)$ and $f^k(z) \rightarrow 0$ if $\mid z \mid < 1$. Therefore

$$A(0) = \{z : \lim_{k \to \infty} R^k(z) = 0\} = \{z : \mid z \mid < 1\}.$$

1.7 The Julia set and the Fatou set

In complex dynamics the Julia set and the Fatou set are two complementary sets defined for a function. The Fatou set of the function consists of points with the property that all nearby points behave similarly under repeated application of the function. The Julia set consists of points such that an arbitrarily small disturbance can cause radical changes in the sequence of iterates. Thus the behavior of the function on the Fatou set is regular. The Julia set of a function f is commonly denoted by $J(f)$, and the Fatou set is denoted by $F(f)$. These sets are named after the French mathematicians Gaston Julia and Pierre Fatou.

Julia set can be defined in many ways. If f is a polynomial $J(f)$ is the clousure of repelling periodic points of f. It can also be defined in terms of the behaviour of the iterates $f^k(z)$ for large k. First we define the filled-in-Julia set for polynomial f.

$$K(f) = \{z \in \mathbb{C} : \lim_{k \to \infty} f^k(z) \neq \infty\}$$

The Julia set of f is the boundary of the filled-in Julia set that means $J(f) = \partial K(f)$. So in every neighbourhood of z there are points w and v with $f^k(w) \to \infty$ and $f^k(v) \nrightarrow \infty$. The complement of the Julia set is called the Fatou set or the stable set. For example, the filled-in Julia set of z^2 is the unit disk $\mid z \mid \leq 1$ and the Julia set is its boundary, the unit circle[2].

1.8 Conjugacy

The rational maps of degree one are the Mobius maps

$$z \mapsto \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

where $a, b, c, d \in \mathbb{C}$ and they can be used to introduce conjugacy. Two rational maps R and S are conjugate if there is some Mobius map g with

$$S = gRg^{-1}.$$

An important property of conjugacy is that it respects iteration. That is, if $S = gRg^{-1}$ then $S^n = gR^ng^{-1}$. This means that we can transfer a problem concerning R to a (possibly simpler) problem concerning a conjugate S of R and then attempt to solve this in terms of S. If we wish, we can rewrite the solution in terms of R. Another useful property of conjugacy is that it respects fixed points: explicitly, if $S = gRg^{-1}$ then S fixes $g(z)$ if and only if R fixes z. In finding Julia set of a rational function conjugacy plays an important role as shown in the following.

Theorem 1.8.1. *[1] Let R be a non-constant rational map, let g be a mobius map and let $S = gRg^{-1}$. Then $J(S) = g(J(R))$.*

Chapter 2

Properties of Newton's method

In this chapter we analyze some important properties of Newton's method. For Newton's method applied to any polynomial f, each root of f is fixed points of N, and that is the only finite fixed points. We call a fixed point w of $N(z)$ extraneous if $f(w) \neq 0$ i.e. w is not a root of f. First, we prove some properties of Newton's method.

2.1 Fixed point properties

Lemma 2.1.1. *[5]Let f be a polynomial. Then $N(z)$ and $N_k(z)$ $k \in \mathbb{N}$ has no extraneous fixed points except infinity.*

Proof. An extraneous fixed-point of $N(z) = z - \frac{f(z)}{f'(z)}$ is a fixed-point different from all the roots of f. Now if θ is a fixed point of N then $N(\theta) = \theta$ and it follows that $f(\theta) = 0$. So $N(z)$ has no extraneous fixed point. For $N_k(z)$ there are also no finite extraneous fixed points. Now, infinity is a fixed pont of $N(z)$ because 0 is a fixed point of its conjugate map $S(z) = gNg^{-1}(z)$, where $g(z) = \frac{1}{z}$. So infinity is the only extraneous fixed point of $N(z)$. The proof for N_k follows similarly. $\square$

Lemma 2.1.2. *[7]. Infinity is the repelling fixed point of $N(z)$.*

Proof. Newton's method for a polynomial $f(z)$ is $N(z) = z - \frac{f(z)}{f'(z)}$. Taking $f(z) = \sum_{i=0}^{n} a_i z_i$ we get

$$\begin{aligned} N(z) &= z - \frac{\sum_{i=0}^{n} a_i z^i}{\sum_{i=0}^{n} i a_i z^{i-1}} \\ &= \frac{\sum_{i=0}^{n} (i-1) a_i z^i}{\sum_{i=0}^{n} i a_i z^{i-1}}. \end{aligned}$$

Putting $\frac{1}{z}$ in place of z we get

$$\begin{aligned} N(\frac{1}{z}) &= \frac{\sum_{i=0}^{n} (i-1) a_i \frac{1}{z^i}}{\sum_{i=1}^{n} i a_i \frac{1}{z^{i-1}}} \\ &= \frac{\sum_{i=0}^{n} (i-1) a_i z^{n-i}}{\sum_{i=0}^{n} i a_i z^{n+1-i}}. \end{aligned}$$

Setting $S(z) = \frac{1}{N(\frac{1}{z})}$ we get $S(0) = 0$. So infinity is a fixed point of N.
Now calculating $S'(z)$ we get

$$S'(0) = \frac{n}{n-1} > 1.$$

Hence infinity is repelling fixed point of $N(z)$. $\square$

Lemma 2.1.3. *[6] The Julia set of $N(z)$ is unbounded.*

Proof. As infinity is a repelling fixed point of $N(z)$, points near infinity would be divergent away from infinity by $N(z)$.
As $J(N(z)) = \partial K(z)$, where $K(z) = \{z \in \mathbf{C} : \lim_{k \to \infty} N^k(z) \neq \infty\}$.
$\infty \in J(N(z))$. Therefore $J(N(z))$ is unbounded. $\square$

Fixed points of Newton's method play an important role in determining its Julia set of it. some of their property as follows.

Lemma 2.1.4. *[5] N(z) has precisely* $d+1$ *fixed points where* d *is the degree of* $N(z)$*, counted with multiplicity.*

Proof. Newton's function for a polynomial $f(z)$ is

$$\begin{aligned} N(z) &= z - \frac{f(z)}{f'(z)} \\ &= \frac{zf'(z) - f(z)}{f'(z)} = \frac{P(z)}{Q(z)} \ \ (say). \end{aligned}$$

Now $deg(N(z)) = max\{degP(z), degQ(z)\}$. Here we consider to cases.
Case 1. Let $d = deg\ Q(z)$. Now if $N(z) = z$, then $zQ(z) - P(z) = 0$
As $zQ(z) - P(z)$ is of degree $d+1$, then $zQ(z) - P(z) = 0$ has d+1 solutions. So $N(z) = z$ has d+1 solutions. In other words $N(z)$ has $d+1$ fixed points.
Case 2. Let $d = deg\ P(z)$. The finite fixed points of $N(z)$ are solutions to $N(z) = z$ i.e. $P(z) - zQ(z) = 0$. This equation may have no solutions, or as many as d solutions. First we consider the case when the equation has no solution. In this case $P(z) - zQ(z)$ must be a nonzero constant c and deg $P(z) = d$, $deg\ Q(z) = d - 1$. So $P(z)$ and $Q(z)$ as follows
$P(z) = a_d z^d + a_{d-1} z^{d-1} + ... + a_0$, $Q(z) = a_d z^{d-1} + a_{d-1} z^{d-2} + ... + a_1$, $c = a_0$.
Here all $a_i \in \mathbb{C}$. We show that 0 is a fixed point of multiplicity $d+1$ in the conjugate map $S(z)$.
Considering a new conjugate map $S(z) = \frac{z^d Q(\frac{1}{z})}{z^d P(\frac{1}{z})}$ we get

$$S(z) = \frac{a_1 z^d + a_2 z^{d-1} + ... + a_d z}{a_0 z^d + a_1 z^{d-1} + ... + a_d}.$$

So $S(0) = 0$ and 0 is a fixed point of multiplicity $d+1$ for the conjugate map

$S(z)$. Hence ∞ is a fixed point of $N(z)$ with multiplicity $d+1$.

Now we consider the case when the equation has $r \geq 1$ solution. Then $P(z) - zQ(z) = 0$ has r solution. So we can take

$P(z) = a_d z^d + a_{d-1} z^{d-1} + ... + a_{r+1} z^{r+1} + a_r z^r + ... + a_1 z + a_0$ and

$Q(z) = a_d z^{d-1} + a_{d-1} z^{d-2} + ... a_{r+1} z^r + b_r z^{r-1} + ... + b_1 z + b_0, \;\; b_r \neq a_r$

where all $a_i, b_i \in \mathbb{C}$. By same process it can be shown that, ∞ is a fixed point of $R(z)$ with multiplicity $(d+1-r)$. Also $P(z) - zQ(z) = 0$ has r solutions. So $N(z)$ has r finite fixed points. Hence total number of fixed points of $N(z)$ is $(d+1-r+r)$ i.e. $(d+1)$. $\square$

Lemma 2.1.5. *Any simple root of a polynomial $f(z)$ is super attracting fixed point of $N(z)$.*

Proof. Newton's function for a polynomial $f(z)$ is

$$N(z) = z - \frac{f(z)}{f'(z)}.$$

Derivating we get $N'(z) = \dfrac{f(z)f''(z)}{[f'(z)]^2}$. Now if θ is a simple root then $N'(\theta) = 0$. So any simple root is super attracting fixed point of $N(z)$. $\square$

Lemma 2.1.6. *Any simple root of $f(z)$ is either repelling or indifferent fixed point of relaxed Newton's method $N_k(z)$ where $k > 1$.*

Proof. Relaxed Newton's function for a polynomial $f(z)$ is

$$N_k(z) = z - k\frac{f(z)}{f'(z)}.$$

Derivating we get $N_k'(z) = (1-k) + k\dfrac{f(z)f''(z)}{[f'(z)]^2}$. Now if θ is a simple root then $\mid N_k'(\theta) \mid = \mid 1-k \mid \geq 1$. So any simple root is either repelling or indifferent fixed point of $N_k(z)$ whenever $k > 1$.

$\square$

Lemma 2.1.7. *Any multiple root of $f(z)$ is an atrracting fixed point of $N(z)$.*

Lemma 2.1.8. *If f is a polynomial with a root of order k and $N_k(z)$ is the corresponding relaxed Newton's method then $N_k(z)$ can have any type of fixed points.*

Proof. Let $f(z) = g(z)(z-b)^m, \;\; g(b) \neq 0$. Then the Relaxed Newton's function for a polynomial $f(z)$ is $N_k(z) = z - k\frac{f(z)}{f'(z)} = z - k\dfrac{g(z)(z-b)}{g'(z)(z-b)+mg(z)}$. Derivating we get $N_k'(z) = 1 - k\left[\dfrac{g(z)(z-b)}{g'(z)(z-b)+mg(z)}\right]'$. Putting $z = b$ we get $N_k'(b) = 1 - \frac{k}{m}$. So the value of $\mid N_k'(b) \mid$ depends on k and m. Hence $N_k(z)$ may have any type of fixed point for multiple roots of $f(z)$. $\square$

We sum up the previous lemmas in two theorems.

Theorem 2.1.9. *Given a polynomial $p(z)$ of degree n, each root is an attractive fixed point of $N(z)$ if it is a multiple root, and a superattractive fixed point if it is a simple root. Moreover, ∞ is a repelling fixed point.*

Theorem 2.1.10. *Given a polynomial $p(z)$ of degree n, a root can be any type of fixed point of $N_m(z)$ if it is a multiple root, and a repelling or indifferent fixed point if it is a simple root. Moreover, ∞ is a repelling fixed point.*

2.2 Connectedness of Julia set of Newton's method

A connected set is a set that cannot be partitioned into two nonempty subsets which are open in the relative topology induced on the set. Equivalently, it is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other. Shishikura and kalantari give some result on connectedness of Julia set of a rational function. Applying these on Newton's method following results are obtained.

Theorem 2.2.1 (**Shishikura**[6]). *If Julia set of a rational function R is not connected, then certainly there are two fixed points of R with eigen values $\lambda = 1$ or $\mid \lambda \mid > 1$.*

Theorem 2.2.2 (**Kalantari (2003)**[5]). *Let θ be a fixed point of Newton's method $N(z) = z - \frac{f(z)}{f'(z)}$ for a polynomial $f(z)$. Then*

$$N'(\theta) = \begin{cases} 0 & \text{if } f(\theta) = 0 \text{ but } f'(\theta) \neq 0, \\ \frac{s-1}{s} & \text{if } \theta \text{ is a root of multiple } s \text{ of } f(z). \end{cases}$$

Theorem 2.2.3. *[6] $J(N(z))$ is connected.*

Proof. Let it be not connected, then by theorem 2.2.1 it should have two fixed points with eigen values $\lambda = 1$ or $\mid \lambda \mid > 1$. But by theorem 2.2.2 it is clear that all fixed point have eigen value 0 or less than 1. So by contradiction Julia set of standard Newton's method is connected. Otherwise we can say it directly from following theorem as ∞ is the only repelling fixed point for $N(z)$. □

We can also easily state it by using another theorem.

Theorem 2.2.4 (**Shishukura(1990)**[5]). *If a rational map has only one fixed point which is repelling, then its Julia set is connected.*

2.3 Periodic points of Newton's method.

Lemma 2.3.1. *[5] Let R be a rational map of degree $d \geq 2$. Suppose θ is a fixed point of R. Moreover, assume that if θ is a rationally indfferent fixed point its multiplier $\lambda \geq 1$. Then for all $n \geq 1$, θ has the same multiplicity in R^n*

Lemma 2.3.2. *$N(z)$ of degree $d \geq 2$ has infinitely many periodic points.*

Proof. First we assume that if θ is any rationally indifferent fixed point of N, then $N'(\theta) = 1$.
Let p be any prime number. We show that N^p has a fixed point different from those of N. Let ϕ be any fixed point of N^p. Now If it is a fixed point of N, then from the previous lemma it must have the same multiplicity in $N(z) - z$ and $N^p(z) - z$.0 But we know $N(z)$ have $d + 1$ fixed points. So multiplicity of ϕ in $N^p(z) - z$ is at most $d + 1$. Again if $N^p(z)$ has no other fixed points than those of N, each having multiplicity at most $(d + 1)$, then $N^p(z)$ would have at most $(d + 1)^2$ fixed points. Also $N^p(z)$ have $d^p + 1$ number of fixed points, but As $d^p + 1$ exceeds $(d + 1)^2$. Hence for a given prime p distinct periodic point should exist.
Since periodic points corresponding to different prime periods must be distinct, this completes the proof of the infnity of periodic point for the rational maps N having only rationally indiferent fixed points with multiplier equal one.
Next we consider the general case of N. If N has finitely many periodic points it has finitely many rationally indifferent periodic points. Suppose that N has t periodic points $\phi_1, \phi_2, ..., \phi_t$ of periods $p_1, p_2, ..., p_t$ with multipliers $\lambda_1, \lambda_2, ..., \lambda_t$ which are respectively , $n_1, n_2, ..., n_t$th root of unity.

Set

$$p = \prod_{i=1}^{t} p_i n_i.$$

Then clearly each ϕ_i is a fixed point of N^p. Moreover from the chain rule $(N^p)'(\phi_i) = 1$. Thus by the above argument N^p must have an infinite number of periodic points and so does N. □

Theorem 2.3.3 (Shishikura [5])**.** *Let R be a rational map of degree $d \geq 2$, then there are at most $2d - 2$ periodic cycles which are either attractive, superattractive, or indifferent.*

Then from this theorem and previous lemmas we have

Corollary 2.3.4. *[5] $N(z)$ of degree $d \geq 2$ has infinitely many repelling periodic points.*

Lemma 2.3.5. *$N(z)$ of degree $d \geq 2$ has at most $2d - 2$ superattractive periodic points.*

Proof. Before proving this we state a formula for periodic points.
If θ is a periodic point of period p, then the set

$$\theta^{(p)} = \theta, R(\theta), R^2(\theta), ..., R^{p-1}(\theta)$$

consists of p distinct elements called, cycle or periodic cycle. Setting $\theta_i = R^i(\theta),\ \ i = 0, 1, ..., p - 1$ we have $R^p(\theta_i)\theta_i$

so that each member on the cycle is a periodic point of R^p. Interestingly, from repeated application of the Chain rule we get

$$(R^p)'(\theta_i) = \prod_{j=0}^{p-1} R'(\theta_j),\ \ \forall i = 0, 1, ..., p - 1.$$

Now we start our main proof. First we show that each superattractive periodic point of N is associates with a unique critical point of N.Let ξ be a superattractive periodic point of N of period p, belonging to a cycle $\xi^{(p)} = \xi_j, j = 1, 2, ..., p$. Since ξ is a super attractive fixed point of N^p, $(R^p)'(\xi) = 0$. But by above formula we get

$$(R^p)'(\xi) = \prod_{j=1}^{p} R'(\xi_j) = 0$$

where $\xi_j, j = 0, 1, ..., p-1$ is the corresponding cycle. Thus $R'(\xi_j) = 0$ for some j. But the cycle containing ξ is unique. Otherwise, ξ is at the intersection of two cycles, hence it must lie on a smaller cycle, contradicting that its period is p. Hence each superattractive periodic point of N is associates with a unique critical point of N.

As a rational map R of degree $d \geq 2$ has $2d-2$ critical points,countingwith multiplicity, we can say that $N(z)$ of degree $d \geq 2$ has at most $2d-2$ superattractive periodic points. $\square$

Chapter 3

Julia set of Newton's method for quadratic polynomial

We are now ready to discuss the Julia set of Newton's method. For the sake of simplicity we start with the quadratic case. The quadratic examples might lead us to investigate the Julia set of this method for higher degree polynomials having degree greater than two.However, for higher degree polynomial the situation is fundamentally different[4]. First we take a polynomial having root at 1 and -1, then gradually we change the roots and see what changes occurs in the Julia sets.

3.1 With roots 1 and -1.

Newton's method for $f(z) = k(z^2 - 1)$, where $k \in \mathbb{C}$ is

$$\begin{aligned} N(z) &= z - \frac{k(z^2-1)}{2kz} \\ &= \frac{1}{2}(z + \frac{1}{z}). \end{aligned}$$

Now roots of $f(z) = 0$ are 1 and -1. So $N(1) = 1$ and $N(-1) = -1$. Also 1 and -1 are super attracting fixed points of $N(z)$. Let the Basin of attraction for 1 and -1 be denoted by $A(1)$and $A(-1)$ respectively. So
$A(1) = \{z : \lim_{k\to\infty} N^k(v) = 1\}$,
$A(-1) = \{z : \lim_{k\to\infty} N^k(v) = -1\}$.
Take the mapping $w = \frac{z+1}{z-1}$. Now $Z_{n+1} = N(z_n) = z_n - \frac{f(z_n)}{f'(z_n)}$. Hence $z_{n+1} = \frac{1}{2}(z_n + \frac{1}{z_n})$. Now putting the value of z_n and z_{n+1} we get, $\frac{w_{n+1}+1}{w_{n+1}-1} = \frac{1}{2}\left(\frac{w_n+1}{w_n-1} + \frac{w_n-1}{w_n+1}\right)$.
This imply that $w_{n+1} = w_n^2 \equiv F(w_n)$ (say). Now $\{w : \lim_{k\to\infty} F^k(w) = 0\} = \{w :| w |< 1\}$ and $\{w : \lim_{k\to\infty} F^k(w) = \infty\} = \{w :| w |> 1\}$.
Now when $w = 0$, $z = -1$. and $w = \infty$, $z = 1$. So the mapping $z = \frac{w+1}{w-1}$ maps the unit disk in w plane to the left half of the z plane and outside of the unit disk in w plane to the right half of the z plane.
Now $z = \frac{w+1}{w-1}$. Taking $z = x+iy$ and $w = a+ib$ we get $x+iy = \frac{a+ib+1}{a+ib-1}$.
This implies that

$$x + iy = \frac{a^2 + b^2 - 1 - 2ib}{(a-1)^2 + b^2}.$$

and consequently

$$x = \frac{a^2 + b^2 - 1}{(a-1)^2 + b^2}.$$

Now when $| w |> 1$, $a^2 + b^2 > 1$ so when $x > 0$ then $Real(z)$ must be greater than zero. So $A(1) = \{z : \lim_{k\to\infty} N^k(v) = 1\} = \{w : \lim_{k\to\infty} F^k(w) = \infty\} = \{w :| w |> 1\} = \{z : \Re(z) > 0\}$. Similarly $A(-1) = \{z : \lim_{k\to\infty} N^k(z) = -1\} = \{w : \lim_{k\to\infty} F^k(w) = 0\} = \{w :| w |< 1\} = \{z : \Re(z) < 0\}$. So $\partial A(1) = \partial A(-1) = \{z : \Re(z) = 0\}$.
Hence Julia set of $N(z)$ is the imaginary axis.

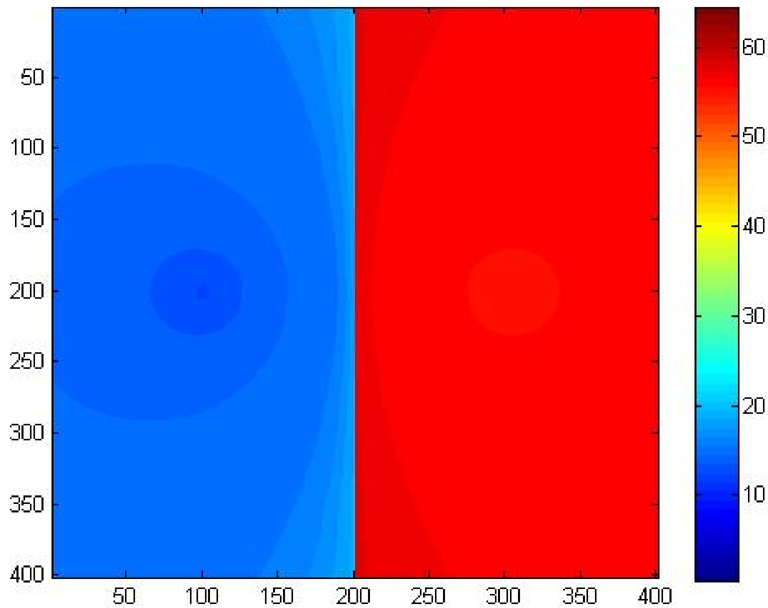

Figure 3.1: Basin of attraction for roots of $z^2 - 1$ via Newton's method.

<u>Matlab program for Julia set of Newton's method for $f(z) = z^2 - 1$</u>

```
[X, Y] =meshgrid([−2 : 0.01 : 2]);
[m, n] =size(X);
maxiters= 30;
T = 64∗ ones(m, n);
%parameters for the function C(z) = z^2 + c
c = −1
p = [1  0  c];
%numerical values for the roots
rts=roots(p);
```

```
for j = 1 : m
fprintf ('on iterate %d\n ',j);
for k = 1 : n
z = X(j,k) + i * Y(j,k);
for t = 1:maxiters
y = z - (z^2 + c)/(2 * z);
z=y;
if abs(z-rts(1)) < .001
T(j,k) = 12 + t * (23/ maxiters);
break;
end
if abs(z-rts(2)) < .001
T(j,k) = 55 + t * (10/ maxiters);
break;
end
end
end
end
end
image(T)
colorbar
```

3.2 With equal and opposite roots[2]

Newton's method for a polynomial$f(z) = k(z^2 - c)$, where $c, k \in \mathbb{C}$ and $c \neq 0$ is

$$\begin{aligned} N(z) &= z - \frac{k(z^2 - c)}{2kz} \\ &= \frac{z^2 + c}{2z}. \end{aligned}$$

Now roots of $f(z) = 0$ are $\sqrt{c}$ and $-\sqrt{c}$. So $N(\sqrt{c}) = \sqrt{c}$ and $N(-\sqrt{c}) = -\sqrt{c}$. Also $\sqrt{c}$ and $-\sqrt{c}$ both are super attracting fixed points of $N(z)$.

Now $N(z) + \sqrt{c} = \dfrac{(z + \sqrt{c})^2}{2z}$ and $N(z) - \sqrt{c} = \dfrac{(z - \sqrt{c})^2}{2z}$.

So

$$\frac{N(z) + \sqrt{c}}{N(z) - \sqrt{c}} = \frac{(z + \sqrt{c})^2}{(z - \sqrt{c})^2}.$$

Now if $\left|\dfrac{z + \sqrt{c}}{z - \sqrt{c}}\right| < 1$ then $\left|\dfrac{N^k(z) + \sqrt{c}}{N^k(z) - \sqrt{c}}\right| \longrightarrow 0 \quad as \quad k \to \infty.$

So $N^k(z) \longrightarrow -\sqrt{c}$.

Also$\left|\dfrac{z + \sqrt{c}}{z - \sqrt{c}}\right| > 1$ then $\left|\dfrac{N^k(z) + \sqrt{c}}{N^k(z) - \sqrt{c}}\right| \longrightarrow \infty \quad as \quad k \to \infty.$

So $N^k(z) \longrightarrow \sqrt{c}$.

Hence for $f(z) = z^2 - c$, the Julia set of $N(z)$ is the line $\mid z + \sqrt{c} \mid = \mid z - \sqrt{c} \mid$.

3.3 With two distinct roots

Newton's method for a polynomial $f(z) = k(z-a)(z-b)$, where $k, a, b \in \mathbb{C}, a \neq b$ is

$$\begin{aligned} N(z) &= z - \frac{k(z-a)(z-b)}{k(2z-a-b)} \\ &= \frac{z^2 - ab}{2z - a - b}. \end{aligned}$$

Roots of $f(z)$ are a and b. So $N(a) = a$ and $N(b) = b$. Also a and b both are super attracting fixed point point of $N(z)$. Let us take the mapping

$$h(z) = \frac{z-a}{z-b}.$$

Then $h^{-1}(z) = \dfrac{bz-a}{z-1}$. Now to deduce $h \circ N \circ h^{-1}(z)$, note that

$$\begin{aligned} N(h^{-1}(z)) &= \frac{\left(\dfrac{bz-a}{z-1}\right)^2 - ab}{2\left(\dfrac{bz-a}{z-1}\right) - a - b} \\ &= \frac{bz^2 - a}{z^2 - 1}. \end{aligned}$$

Therefore,

$$h \circ N \circ h^{-1}(z) = \frac{\left(\dfrac{bz^2-a}{z^2-1}\right) - b}{\left(\dfrac{bz^2-a}{z^2-1}\right) - a} = z^2 \quad \equiv P(z) \;\; (say).$$

As $h \circ N \circ h^{-1}(z) = P(z)$,Julia sat of $(P(z)) = h($Julia set of $(N(z)))$. i.e. $J(N) = h^{-1}(J(P)) = h^{-1}(J(z^2))$.

Hence $J(N) = h^{-1}(z :\mid z \mid = 1)$ where $h(z) = \dfrac{z-a}{z-b}$.

3.4 For two equal roots

Newton's method for a polynomial $f(z) = k(z-a)^2$, where $k, a \in \mathbb{C}$ is

$$\begin{aligned} N(z) &= z - \frac{k(z-a)^2}{2k(z-a)} \\ &= \frac{z+a}{2}. \end{aligned}$$

Only root of $f(z)$ is a. So $N(a) = a$. Also a is attracting fixed point point of $N(z)$. Let us take the mapping $h(z) = z - a$. So $h^{-1}(z) = z + a$. Now we deduce $h \circ N \circ h^{-1}(z)$.
$N(h^{-1}(z)) = \dfrac{z}{2} + a$. So $h \circ N \circ h^{-1}(z) = \dfrac{z}{2} = P(z)$(say).
As $h \circ N \circ h^{-1}(z) = P(z)$.
Therefore $J(N) = h^{-1}(J(P)) = h^{-1}(J(\frac{z}{2}))$.
Hence $J(N) = h^{-1}(\phi)$ where $h(z) = z - a$ and ϕ is a null set.

In previous example order of root of the polynomial is two. We apply standard Newton method. As the method converges linearly thats why we should apply relaxed Newton method $N_2(z)$which converges quadratically to a double root.

3.5 Relaxed Newton's method for a quadratic polynomial having two equal roots

Relaxed Newton's method of order 2 for a polynomial $f(z) = k(z-a)^2$, where $k, a \in \mathbb{C}$ is

$$\begin{aligned} N_2(z) &= z - \frac{2k(z-a)^2}{2k(z-a)} \\ &= a. \end{aligned}$$

Only root of $f(z) = 0$ is a. So $N_2(a) = a$. Also a is super attracting fixed point point of $N(z)$. Let us take the mapping

$$h(z) = z - a.$$

So $h^{-1}(z) = z + a$. Now we deduce $h \circ N \circ h^{-1}(z)$.
$N(h^{-1}(z)) = a$. So $h \circ N \circ h^{-1}(z) = 0 = P(z)$(say).
As $h \circ N \circ h^{-1}(z) = P(z)$.
So $J(N) = h^{-1}(J(P)) = h^{-1}(J(0))$.
Hence $J(N) = h^{-1}(\phi)$ where $h(z) = z - a$ and ϕ is a null set.

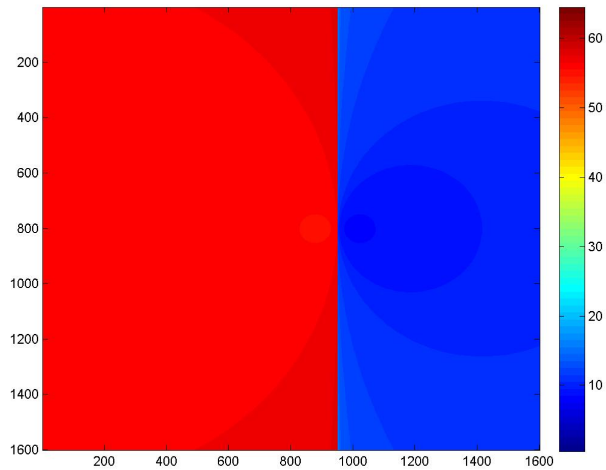

Figure 3.2: Basin of attraction for roots of $(z-1)(z-2)$ via Newton's method.

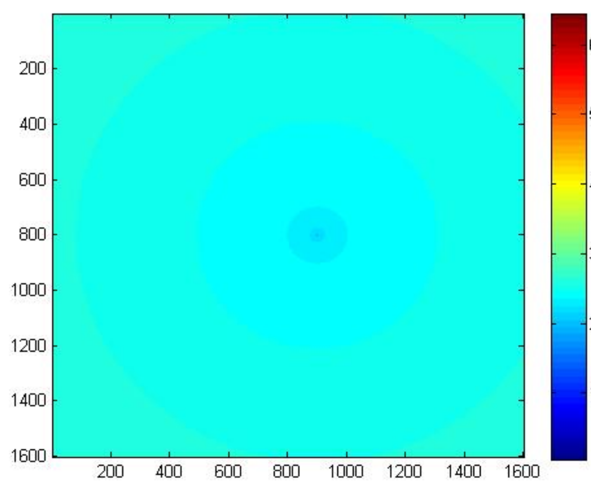

Figure 3.3: Basin of attraction for roots of $(z-1)^2$ via Newton's method

Chapter 4

Julia set of Relaxed Newton's method for polynomials with two roots

The analysis of Newton's method becomes more complicated when degree of polynomial is greater than two. Here we consider polynomials having two distinct roots where at least one root is multiple and move to find Julia set of Newton's function for this. In [3] W. J. Gilbert gives a complete description for the Relaxed Newton's method for a double root applied to a cubic polynomial, and show that the dynamics is conjugate to that of a well-known Julia set which is julia set of quadratics. There is a similar result for the relaxed Newton's method for a root of order m applied to a polynomial of degree $(m+1)$. We show that the Julia sets, applied to any degree polynomial with two distinct roots, are conjugate to the Julia set of quadratics.

4.1 For a cubic polynomial with a double root

Theorem 4.1.1. *[3] The relaxed Newton's method, N_2, applied to the cubic polynomial $k(z-a)^2(z-b)$ where $a, b, k \in \mathbb{C}$, $a \neq b$, is conjugate to $z^2 - \dfrac{3}{4}$.*

Proof. The relaxed Newton's method, N_2, applied to $f(z) = k(z-a)^2(z-b)$ is

$$
\begin{aligned}
N_2(z) &= z - 2\frac{f(z)}{f'(z)} \\
&= z - 2\frac{k(z-a)^2(z-b)}{[k(z-a)^2(z-b)]'} \\
&= z - 2\frac{k(z-a)^2(z-b)}{k[2(z-a)(z-b)+(z-a)^2]} \\
&= z - 2\frac{(z-a)(z-b)}{[2(z-b)+(z-a)]} \\
&= \frac{z^2+az-2ab}{3z-2b-a}.
\end{aligned}
$$

Now we find the critical points of $N_2(z)$ by solving $N_2'(z) = 0$.
$N_2'(z) = 0$, implies that $\left[\dfrac{z^2+az-2ab}{3z-2b-a}\right]' = 0$. So
$\dfrac{(3z-2b-a)(2z+a)-3(z^2+az-2ab)}{(3z-2b-a)^2} = 0.$
Calculating we get, $\dfrac{(z-a)(3z+a-4b)}{(3z-2b-a)^2} = 0.$
Hence $z = a$ and $z = \dfrac{4b-a}{3}$ are critical points of $N_2(z)$.
We know that critical points of the quadratic $p(z) = z^2 + c$ on the Riemann sphere are $z = 0$ and $z = \infty$.
Let us take a mapping

$$h(z) = \frac{3z-4b+a}{2(z-a)}.$$

We see that h maps the critical point of $N_2(z)$ to those of quadratic $p(z) = z^2 + c$.

Now we deduce $h \circ N_2 \circ h^{-1}(z)$.

We have $h^{-1}(z) = \dfrac{4b - a - 2az}{3 - 2z}$. Then

$$\begin{aligned} N_2 h^{-1}(z) &= \frac{\{h^{-1}(z)\}^2 + a\{h^{-1}(z)\} - 2ab}{3\{h^{-1}(z)\} - 2b - a} \\ &= \frac{\{\frac{4b-a-2az}{3-2z}\}^2 + a\{\frac{4b-a-2az}{3-2z}\} - 2ab}{3\{\frac{4b-a-2az}{3-2z}\} - 2b - a}. \end{aligned}$$

After multiplying by $\{3 - 2z\}^2$, the numerator becomes

$(4b - a - 2az)^2 + a(4b - a - 2az)(3 - 2z) - 2ab(3 - 2z)^2$.

Simplifying we get

$8az^2(a - b) + 16b^2 - 14ab - 2a^2$. And the denumerator becomes

$3(4b - a - 2az)(3 - 2z) - (a + 2b)(3 - 2z)^2$.

Simplifying we get $8z^2(a - b) + 18b - 18a$.

Now

$$\begin{aligned} N_2 h^{-1}(z) &= \frac{8az^2(a-b) + (a-b)(-2a-16b)}{8z^2(a-b) + 18b - 18a} \\ &= \frac{4az^2 - (a + 8b)}{4z^2 - 9} \end{aligned}$$

and

$$
\begin{aligned}
h \circ N_2 \circ h^{-1}(z) &= \frac{3N_2 h^{-1}(z) - 4b + a}{2(N_2 h^{-1}(z) - a)} \\
&= \frac{3\dfrac{4az^2 - (a+8b)}{4z^2 - 9} - 4b + a}{2(\dfrac{4az^2 - (a+8b)}{4z^2 - 9} - a)} \\
&= \frac{3\{4az^2 - (a+8b)\} + (a-4b)(4z^2-9)}{2\{4az^2 - (a+8b) - (4z^2-9)a\}} \\
&= \frac{16z^2(a-b) - 12(a-b)}{16(a-b)} \\
&= z^2 - \frac{3}{4} \ \ (say).
\end{aligned}
$$

This completes the proof. $\square$

<u>Matlab program for Julia set of relaxed Newton's method for $f(z) = (z-1)^2(z+1)$</u>

```
[X, Y] =meshgrid([−4 : 0.01 : 4]);
[m, n] =size(X);
maxiters= 40;
T = 64∗ ones(m, n);
%parameters for the function C(z) = z^3 − z^2 − z + 1
p = [1   −1   −1   1];
%numerical values for the roots
rts=roots(p);
for j = 1 : m
fprintf ('on iterate %d\n ',j);
for k = 1 : n
z = X(j, k) + i ∗ Y(j, k);
```

```
for t = 1:maxiters
y = z - 2 * (z^3 - z^2 - z + 1)/3z^2 - 2 * z - 1);
z=y;
if abs(z-rts(1)) < .001
T(j,k) = 1 + t * (23/ maxiters);
break;
end
if abs(z-rts(2)) < .001
T(j,k) = 30 + t * (10/ maxiters);
break;
end
end
end
end
image(T)
colorbar
```

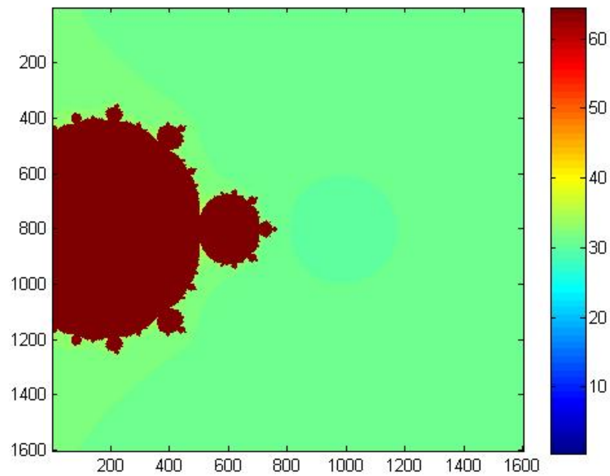

Figure 4.1: Basin of attraction for roots of $(z-1)^2(z+1)$ with respect to $N_2(z)$

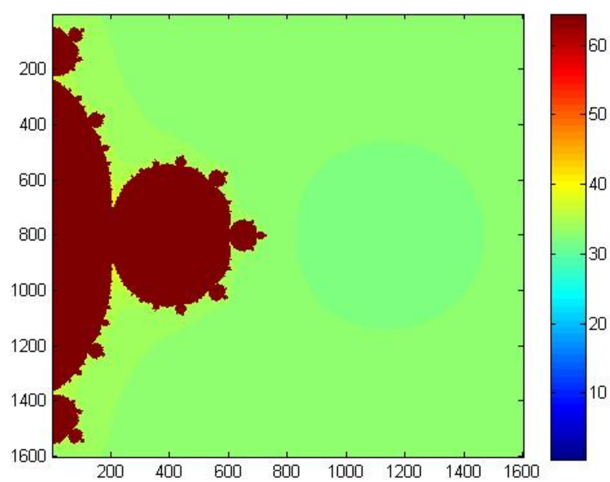

Figure 4.2: Basin of attraction for roots of $(z-2)^2(z+2)$ with respect to $N_2(z)$

4.2 Relaxed Newton's method for a polynomial, with two roots, of degree $(m+1)$, with a root of order m

Theorem 4.2.1. *The relaxed Newton's method, N_m, applied to any polynomial of the type $k(z-a)^m(z-b)$ where $a, b, k \in \mathbb{C}$, $a \neq b$, is conjugate to the quadratic polynomial $z^2 + \dfrac{1-m^2}{4}$.*

Proof. The relaxed Newton's method, N_m, applied to $f(z) = k(z-a)^m(z-b)$ is

$$\begin{aligned}
N_m(z) &= z - m\frac{f(z)}{f'(z)} \\
&= z - m\frac{k(z-a)^m(z-b)}{[k(z-a)^m(z-b)]'} \\
&= z - m\frac{k(z-a)^m(z-b)}{k[m(z-a)^{m-1}(z-b) + (z-a)^m]} \\
&= z - m\frac{(z-a)(z-b)}{[m(z-b) + (z-a)]} \\
&= \frac{z^2 + (m-1)az - abm}{(m+1)z - mb - a}.
\end{aligned}$$

Now we find the critical points of $N_m(z)$ by solving $N_m'(z) = 0$.
$N_m'(z) = 0$ implies that
$\left[\dfrac{z^2 + (m-1)az - abm}{(m+1)z - mb - a}\right]' = 0$. So
$\dfrac{\{(m+1)z - mb - a\}\{2z + (m-1)a\} - \{z^2 + (m-1)az - abm\}(m+1)}{\{(m+1)z - mb - a\}^2} = 0.$

Calculating we get,

$(m+1)z^2 - 2(mb+a)z + (a - am + 2bm) = 0$.
So $(z-a)\{(m+1)z - (a - am + 2bm)\} = 0$.
Hence $z = a$ and $z = \dfrac{a - am + 2bm}{m+1}$ are critical points of $N_m(z)$.
We know that critical points of the quadratic $p(z) = z^2 + c$ on the Riemann sphere are $z = 0$ and $z = \infty$.
Let us take a mapping

$$h(z) = \frac{(m+1)z - (a - am + 2bm)}{m(z-a)}.$$

We see that h maps the critical point of $N_m(z)$ to those of quadratic $p(z) = z^2 + c$.
Now we deduce $h \circ N_m \circ h^{-1}(z)$.
We have $h^{-1}(z) = \dfrac{(a - am + 2bm) - maz}{(m+1) - mz}$. Now

$$\begin{aligned} N_m h^{-1}(z) &= \frac{\{h^{-1}(z)\}^2 + (m-1)ah^{-1}(z) - abm}{(m+1)h^{-1}(z) - mb - a} \\ &= \frac{\{\frac{(a-am+2bm)-maz}{(m+1)-mz}\}^2 + (m-1)a\frac{(a-am+2bm)-maz}{(m+1)-mz} - abm}{(m+1)\frac{(a-am+2bm)-maz}{(m+1)-mz} - mb - a}, \end{aligned}$$

after multiplying by $\{(m+1) - mz\}^2$ the numerator becomes
$\{(a - am + 2bm) - maz\}^2 + (m-1)a\{(a - am + 2bm) - maz\}\{(m+1) - mz\} - abm\{(m+1) - mz\}^2$
$= [m^2a^2z^2 - 2am(a - am + 2bm)z + k^2(a - am + 2bm)^2] + (m-1)a[m^2az^2 + (m+1)(a - am + 2bm) - m(a - am + 2bm)z - amn(m+1)z] - abm[n^2(m+1)^2 + m^2z^2 - 2m(m+1)z]$
$= am^3(a-b)z^2 - m(a-b)[a(1-m)^2 + 4bm]$. And the denumerator becomes

$(m+1)\{(a-am+2bm)-maz\}\{(m+1)-mz\}-(mb+a)\{(m+1)-mz\}^2$
$= k^2(m+1)^2(a-am+2bm)+am^2(m+1)z^2-(mb+a)(m+1)^2-m^2(mb+a)z^2$
$= m^3(a-b)z^2-m(a-b)(1+m)^2.$
So

$$\begin{aligned} N_m h^{-1}(z) &= \frac{am^3(a-b)z^2-m(a-b)[a(1-m)^2+4bm]}{m^3(a-b)z^2-m(a-b)(1+m)^2} \\ &= \frac{am^2z^2-[a(1-m)^2+4bm]}{m^2z^2-(1+m)^2}. \end{aligned}$$

Now

$$\begin{aligned} h\circ N_m\circ h^{-1}(z) &= \frac{(m+1)N_m h^{-1}(z)-(a-am+2bm)}{m(N_m h^{-1}(z)-a)} \\ &= \frac{(m+1)\dfrac{am^2z^2-[a(1-m)^2+4bm]}{m^2z^2-(1+m)^2}-(a-am+2bm)}{m\left(\dfrac{am^2z^2-[a(1-m)^2+4bm]}{m^2z^2-(1+m)^2}-a\right)}. \end{aligned}$$

after multiplying by $m^2z^2-(1+m)^2$ the numerator becomes
$(m+1)\{am^2z^2-[a(1-m)^2+4bm]\}-(a-am+2bm)(m^2z^2-(1+m)^2)$
$= 2m^3(a-b)z^2-(m+1)(2am^2-2am+2bm-2bm^2)$ and the denumerator becomes
$m\{am^2z^2-[a(1-m)^2+4bm]-a(m^2z^2-(1+m)^2)\}$
$= m(4am-4bm).$
So

$$\begin{aligned} hN_m h^{-1}(z) &= \frac{2m^3(a-b)z^2-(m+1)(2am^2-2am+2bm-2bm^2)}{m(4am-4bm)} \\ &= \frac{2m^3(a-b)z^2-2m(m^2-1)(a-b)}{4m^2(a-b)} \\ &= \frac{m}{2}z^2-(\frac{m}{2}-\frac{1}{2m})\equiv q(z)\ \ (say). \end{aligned}$$

Now taking a new map

$$L(z) = \frac{m}{2}z.$$

we observe that

$$L^{-1}(z) = \frac{2}{m}z.$$

Now

$$q(L^{-1}(z)) = \frac{m}{2}(L^{-1}(z))^2 - (\frac{m}{2} - \frac{1}{2m}).$$

Putting the value of $L^{-1}(z)$ and simplifying we get,

$$q(L^{-1}(z)) = \frac{2}{m}z^2 - (\frac{m}{2} - \frac{1}{2m}).$$

Then

$$L(q(L^{-1}(z)) = \frac{m}{2}[\frac{2}{m}z^2 - (\frac{m}{2} - \frac{1}{2m})].$$

Hence

$$L(q(L^{-1}(z))) = z^2 + \frac{1 - m^2}{4}.$$

This completes the proof. $\square$

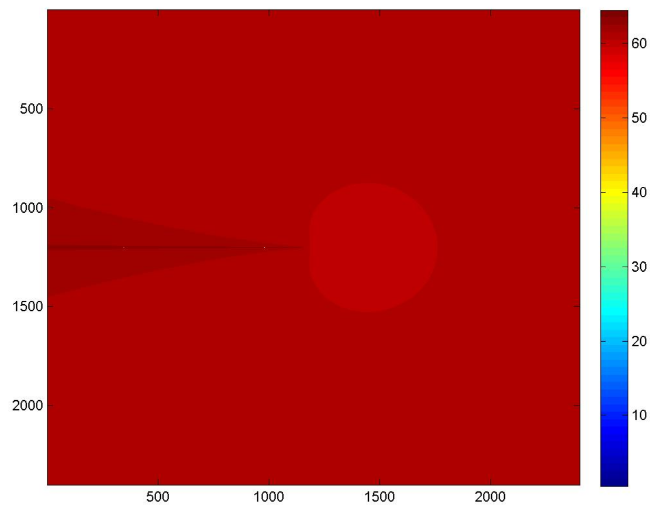

Figure 4.3: Basin of attraction for roots of $(z-1)^3(z+1)$ with respect to $N_3(z)$

4.3 $N_{m-1}(z)$ applied to polynomials with two distinct roots

Theorem 4.3.1. *The relaxed Newton's method, N_{m-1}, applied to a polynomial of type $k(z-a)^{m-1}(z-b)^2$ where $a, b, k \in \mathbb{C}$, $a \neq b$, is conjugate to the quadratic polynomial $z^2 - \dfrac{(m-3)(m+1)}{16}$.*

Proof. The relaxed Newton's method, N_{m-1}, applied to $f(z) = k(z-a)^{m-1}(z-$

$b)^2$ is

$$\begin{aligned}
N_{m-1}(z) &= z-(m-1)\frac{f(z)}{f'(z)}\\
&= z-(m-1)\frac{k(z-a)^{m-1}(z-b)^2}{[k(z-a)^{m-1}(z-b)^2]'}\\
&= z-(m-1)\frac{k(z-a)^{m-1}(z-b)^2}{k[(m-1)(z-a)^{m-2}(z-b)^2+2(z-a)^{m-1}(z-b)]}\\
&= z-(m-1)\frac{(z-a)(z-b)}{[(m-1)(z-b)+2(z-a)]}\\
&= \frac{2z^2+(m-3)az-ab(m-1)}{(m+1)z-(m-1)b-2a}.
\end{aligned}$$

Now we find the critical points of $N_{m-1}(z)$ by solving $N'_{m-1}(z)=0$.

$N'_{m-1}(z)=0$, implies that

$\left[\dfrac{2z^2+(m-3)az-ab(m-1)}{(m+1)z-(m-1)b-2a}\right]'=0.$

Calculating we get, $(m+1)z^2+2(b-mb-2a)z+(3a-2b+2bm-am)=0$.

So $(z-a)\{(m+1)z-(3a-2b-am+2bm)\}=0$.

Hence $z=a$ and $z=\dfrac{3a-2b-am+2bm}{m+1}$ are critical points of $N_{m-1}(z)$.

We know that critical points of the quadratic $p(z)=z^2+c$ on the Riemann sphere are $z=0$ and $z=\infty$.

Let us take a mapping

$$h(z)=\frac{(m+1)z-(3a-2b-am+2bm)}{(m-1)(z-a)}$$

We see that h maps the critical point of $N_{m-1}(z)$ to those of quadratic $p(z)=z^2+c$.

Now we deduce $h\circ N_{m-1}\circ h^{-1}(z)$.

we have $h^{-1}(z)=\dfrac{(3a-2b-am+2bm)-(m-1)az}{(m+1)-(m-1)z}$ Then

$$N_{m-1}h^{-1}(z) = \frac{2\{h^{-1}(z)\}^2 + (m-3)ah^{-1}(z) - ab(m-1)}{(m+1)h^{-1}(z) - (m-1)b - 2a}.$$

Putting the value of $h^{-1}(z)$ and multiplying by $\{(m+1) - (m-1z)\}^2$ the numerator becomes

$2\{(3a - 2b - am + 2bm) - (m-1)az\}^2 + (m-3)a\{(3a - 2b - am + 2bm) - (m-1)az\}\{(m+1) - (m-1z)\} - ab(m-1)\{(m+1) - (m-1z)\}^2$,

calculating we get

$a(m-1)^3a(a-b)z^2 + (m-3)^2(1-m)a^2 + 8(m-1)^2b^2 + (m-1)(m^2 - 14m + 17)ab.$

And the denumerator becomes

$(m+1)\{(3a - 2b - am + 2bm) - (m-1)az\}\{(m+1) - (m-1z)\} - \{(m-1)b - 2a\}\{(m+1) - (m-1z)\}^2$,

calculating we get

$= (m-1)^3(a-b)z^2 - (m+1)^2a(m-1) + b(m-1)(m+1)^2.$

So taking both of them and dividing by $(m-1)$, we get

$$N_{m-1}h^{-1}(z) = \frac{a(m-1)^2a(a-b)z^2 - (m-3)^2a^2 + 8(m-1)b^2 + (m^2 - 14m + 17)ab}{(m-1)^2(a-b)z^2 - (m+1)^2a + b(m+1)^2}.$$

Now $h \circ N_{m-1} \circ h^{-1}(z)$

$$= \frac{(m+1)N_{m-1}h^{-1}(z) - (3a - 2b - am + 2bm)}{(m-1)(N_{m-1}h^{-1}(z) - a)},$$

putting the value of $N_{m-1}h^{-1}(z)$ and after multiplying by $(m-1)^2(a-b)z^2 - (m+1)^2a + b(m+1)^2$ the numerator becomes

$(m+1)\{a(m-1)^2a(a-b)z^2 - (m-3)^2a^2 + 8(m-1)b^2 + (m^2 - 14m + 17)ab\} - (3a - 2b - am + 2bm)\{(m-1)^2(a-b)z^2 - (m+1)^2a + b(m+1)^2\}$

$= 2(m-1)^3(a-b)^2z^2 - 2(m-3)(m^2-1)(a-b)^2.$

And the denumerator becomes

$(m-1)\{a(m-1)^2a(a-b)z^2 - (m-3)^2a^2 + 8(m-1)b^2 + (m^2 - 14m + 17)ab\} - a\{(m-1)^2(a-b)z^2 - (m+1)^2a + b(m+1)^2\})$

$= 8(m-1)^2(a-b)^2.$

So

$$
\begin{aligned}
hN_{m-1}h^{-1}(z) &= \frac{2(m-1)^3(a-b)^2z^2 - 2(m-3)(m^2-1)(a-b)^2}{8(m-1)^2(a-b)^2} \\
&= \frac{m-1}{4}z^2 - \frac{1}{4}\frac{(m-3)(m+1)}{m-1} \equiv q(z) \ \ (say).
\end{aligned}
$$

Now taking a new map

$$L(z) = \frac{m-1}{4}z.$$

We observe that

$$L^{-1}(z) = \frac{4}{m-1}z.$$

$$q(L^{-1}(z)) = \frac{m-1}{4}(L^{-1}(z))^2 - \frac{1}{4}\frac{(m-3)(m+1)}{m-1}.$$

Putting the value of $L^{-1}(z)$ and simplifying we get,

$$q(L^{-1}(z)) = \frac{4}{m-1}z^2 - \frac{1}{4}\frac{(m-3)(m+1)}{m-1}.$$

Now

$$L(q(L^{-1}(z)) = \frac{m-1}{4}[\frac{4}{m-1}z^2 - \frac{1}{4}\frac{(m-3)(m+1)}{m-1}].$$

Hence

$$L(q(L^{-1}(z))) = z^2 - \frac{(m-3)(m+1)}{16}.$$

this completes the proof. □

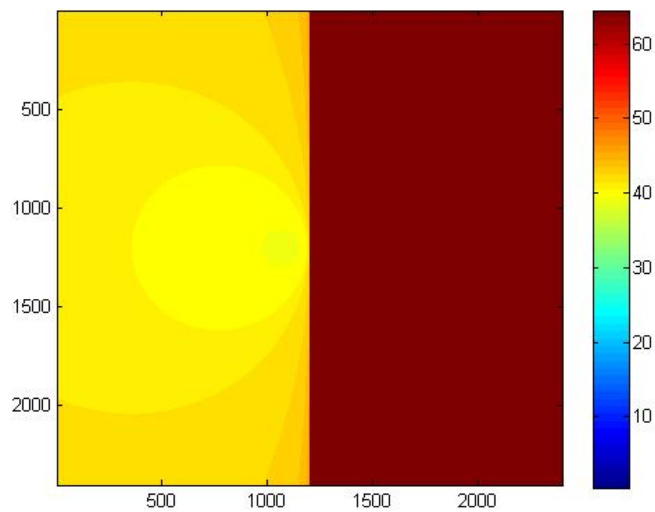

Figure 4.4: Basin of attraction for roots of $(z-1)^2(z+1)^2$ with respect to $N_2(z)$

We now try to generalize it for any degree polynomial having two roots.

4.4 For any polynomial with two distinct roots.

We now generalize the results of this chapter.

Theorem 4.4.1. *The relaxed Newton's method, N_m, applied to any polynomial $k(z-a)^m(z-b)^n$ where $a, b, k \in \mathbb{C}$, $a \neq b$, is conjugate to the quadratic polynomial $z^2 - \frac{1}{4}\left(\frac{m^2}{n^2} - 1\right)$.*

Proof. The relaxed Newton's method, N_m, applied to $f(z) = k(z-a)^m(z-b)^n$

is

$$
\begin{aligned}
N_m(z) &== z - m\frac{f(z)}{f'(z)} \\
&= z - m\frac{k(z-a)^m(z-b)^n}{[k(z-a)^m(z-b)^n]'} \\
&= z - m\frac{k(z-a)^m(z-b)^n}{k[m(z-a)^{m-1}(z-b)^n + n(z-a)^m(z-b)^{n-1}]} \\
&= z - m\frac{(z-a)(z-b)}{[m(z-b)+n(z-a)]} \\
&= \frac{nz^2 + (m-n)az - abm}{(m+n)z - mb - na}.
\end{aligned}
$$

Now we find the critical points of $N_m(z)$ by solving $N_m'(z) = 0$.

$N_m'(z) = 0$ this imply

$\left[\frac{nz^2 + (m-n)az - abm}{(m+n)z - mb - na}\right]' = 0$. So

$\frac{\{(m+n)z - mb - na\}\{2nz + (m-n)a\} - \{nz^2 + (m-n)az - abm\}(m+n)}{\{(m+n)z - mb - na\}^2} = 0$.

Calculating we get, $n(m+n)z^2 - 2n(mb+na)z + (an^2) - anm + 2bmn) = 0$.

So $(z-a)\{n(m+n)z - (an^2 - anm + 2bmn)\} = 0$

Hence $z = a$ and $z = \dfrac{an^2 - anm + 2bmn}{n(m+n)}$ are critical points of $N_m(z)$.

We know the critical points of the quadratic $p(z) = z^2 + c$ on the Riemann sphere are $z = 0$ and $z = \infty$.

Let us take a mapping

$$
h(z) = \frac{n(m+n)z - (an^2 - anm + 2bmn)}{m(z-a)}
$$

We see that h maps the critical point of $N_m(z)$ to those of quadratic $p(z) = z^2 + c$.

Now we deduce $h \circ N_m \circ h^{-1}(z)$.
$h^{-1}(z) = \dfrac{n(an - am + 2bm) - maz}{n(m+n) - mz}$. Then

$$
\begin{aligned}
N_m h^{-1}(z) &= \frac{n\{h^{-1}(z)\}^2 + (m-n)ah^{-1}(z) - abm}{(m+n)h^{-1}(z) - mb - na} \\
&= \frac{n\{\frac{n(an-am+2bm)-maz}{n(m+n)-mz}\}^2 + (m-n)a\frac{n(an-am+2bm)-maz}{n(m+n)-mz}(z) - abm}{(m+n)\frac{n(an-am+2bm)-maz}{n(m+n)-mz} - mb - na}.
\end{aligned}
$$

After multiplying by $\{n(m+n) - mz\}^2$, the numerator becomes
$n\{n(an-am+2bm)-maz\}^2 + (m-n)a\{n(an-am+2bm)-maz\}\{n(m+n)-mz\} - abm\{n(m+n)-mz\}^2$,
$= n[m^2a^2z^2 - 2amn(an-am+2bm)z + k^2(an-am+2bm)^2] + (m-n)a[m^2az^2 + n^2(m+n)(an-am+2bm) - mn(an-am+2bm)z - amn(m+n)z] - abm[n^2(m+n)^2 + m^2z^2 - 2nm(m+n)z]$
$= am^3(a-b)z^2 - mn^2(a-b)[a(n-m)^2 + 4bmn]$.
And the denumerator becomes
$(m+n)\{n(an-am+2bm)-maz\}\{n(m+n)-mz\} - (mb+na)\{n(m+n)-mz\}^2$
$= k^2(m+n)^2(an-am+2bm) + am^2(m+n)z^2 - (mb+na)n^2(m+n)^2 - m^2(mb+na)z^2$
$= m^3(a-b)z^2 - mn^2(a-b)(n+m)^2$.
So

$$
\begin{aligned}
N_m h^{-1}(z) &= \frac{am^3(a-b)z^2 - mn^2(a-b)[a(n-m)^2 + 4bmn]}{m^3(a-b)z^2 - mn^2(a-b)(n+m)^2} \\
&= \frac{am^2z^2 - n^2[a(n-m)^2 + 4bmn]}{m^2z^2 - n^2(n+m)^2}.
\end{aligned}
$$

Now $h \circ N_m \circ h^{-1}(z)$

$$= \frac{n(m+n)N_m h^{-1}(z) - (an^2 - anm + 2bmn)}{m(N_m h^{-1}(z) - a)}.$$

$$= \frac{n(m+n)\dfrac{am^2z^2 - n^2[a(n-m)^2 + 4bmn]}{m^2z^2 - n^2(n+m)^2} - (an^2 - anm + 2bmn)}{m(\dfrac{am^2z^2 - n^2[a(n-m)^2 + 4bmn]}{m^2z^2 - n^2(n+m)^2} - a)}.$$

After multiplying by $m^2z^2 - n^2(n+m)^2$ the numerator becomes

$n(m+n)\{am^2z^2 - n^2[a(n-m)^2 + 4bmn]\} - (an^2 - anm + 2bmn)(m^2z^2 - n^2(n+m)^2)$

$= 2m^3n(a-b)z^2 - n^2(m+n)(2am^2n - 2amn^2 + 2bmn^2 - 2bm^2n).$

And the denumerator becomes

$m\{am^2z^2 - n^2[a(n-m)^2 + 4bmn] - a(m^2z^2 - n^2(n+m)^2)\}$

$= m(4amn^3 - 4bmn^3).$

So

$$\begin{aligned} hN_m h^{-1}(z) &= \frac{2m^3n(a-b)z^2 - n^2(m+n)(2am^2n - 2amn^2 + 2bmn^2 - 2bm^2n)}{m(4amn^3 - 4bmn^3)} \\ &= \frac{2m^3n(a-b)z^2 - 2mn^3(m^2 - n^2)(a-b)}{4m^2n^3(a-b)} \\ &= \frac{m}{2n^2}z^2 - (\frac{m}{2} - \frac{n^2}{2m}) \equiv q(z). \end{aligned}$$

Now taking a new map

$$L(z) = \frac{m}{2n^2}z.$$

We observe that

$$L^{-1}(z) = \frac{2n^2}{m}z.$$

Now

$$q(L^{-1}(z)) = \frac{m}{2n^2}(L^{-1}(z))^2 - (\frac{m}{2} - \frac{n^2}{2m}),$$

putting the value of $L^{-1}(z)$ and simplifying we get,

$$q(L^{-1}(z)) = \frac{2n^2}{m}z^2 - (\frac{m}{2} - \frac{n^2}{2m}).$$

Now

$$L(q(L^{-1}(z)) = \frac{m}{2n^2}[\frac{2n^2}{m}z^2 - (\frac{m}{2} - \frac{n^2}{2m})].$$

Hence

$$L(q(L^{-1}(z))) = z^2 - \frac{1}{4}(\frac{m^2}{n^2} - 1).$$

This completes the proof.

$\square$

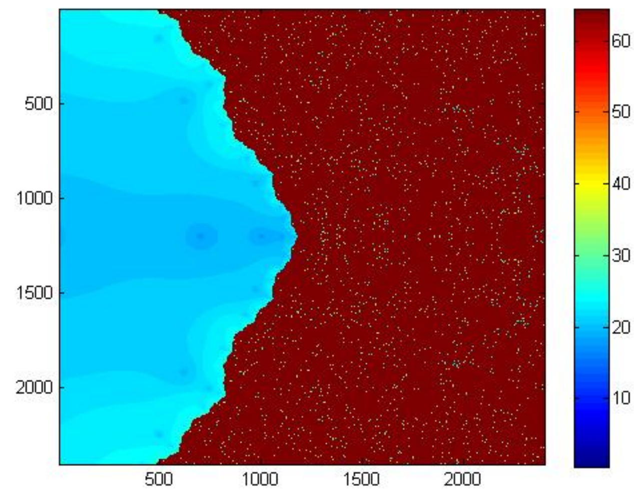

Figure 4.5: Basin of attraction for roots of $(z-1)^3(z+1)^2$ with respect to $N_3(z)$

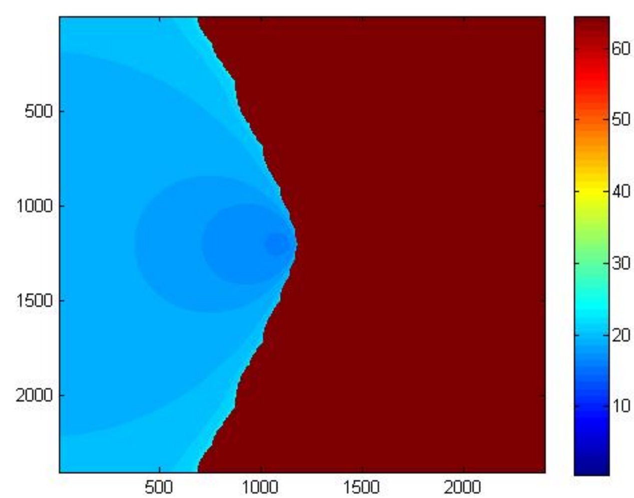

Figure 4.6: Basin of attraction for roots of $(z-1)^3(z+1)^2$ with respect to $N_2(z)$

Chapter 5

Conclusion and future work

Any multiple root is attracting for standard Newton's method where Relaxed Newton's method may have any type of fixed points depending upon the degree of polynomial and the multiplicities of roots. Julia set have a significant role in the dynamical system associated with Newton's method. $J(N)$ is connected but $J(N_m)$ may or may not be connected. For quadratic polynomial with distinct roots, standard Newton's method is globally analytically conjugate to the quadratic polynomial $z \mapsto z^2$. There is a similar result for relaxed Newton's method for polynomials having two distinct roots i.e. Julia set of the relaxed Newton's method applied to the family of polynomials $k(z-a)^m(z-b)^n$, is conjugate to the Julia set of quadratics. The Julia set of relaxed Newton's method considered in chapter 4 can be studied in more details at a later stage. We can study the conjugacy of Julia set of relaxed Newton's method for polynomials having three or more distinct roots.

Bibliography

[1] Alan F Beardon. *Iteration of rational functions: Complex analytic dynamical systems*, volume 132. Springer Science & Business Media, 2000.

[2] Kenneth Falconer. *Fractal geometry: mathematical foundations and applications*. John Wiley & Sons, 2004.

[3] William J Gilbert. The complex dynamics of newton's method for a double root. *Computers & Mathematics with Applications*, 22(10):115–119, 1991.

[4] William J Gilbert. Newton's method for multiple roots. 1994.

[5] Bahman Kalantari. *Polynomial root-finding and polynomiography*. World Scientific, 2009.

[6] Xingyuan Wang and Xuejing Yu. Julia sets for the standard newton's method, halley's method, and schröder's method. *Applied mathematics and computation*, 189(2):1186–1195, 2007.

[7] Wang Xingyuan and Liu Wei. The julia set of newton's method for multiple root. *Applied mathematics and computation*, 172(1):101–110, 2006.